Estimating Parameters in Discrete Time Series

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Abstract

The goal of this work is to estimate the values of unknown parameters $\mathbf{p}$ in discrete dynamical system models given the data points $\mathbf{x}^{(n)}$ and the basic form of the model functions $\mathbf{f}$. The technique could be extended to determining the model $\mathbf{x}^{(n)} = \mathbf{f}(\mathbf{x}^{(n-1)}; \mathbf{p})$, through trials. The technique would be most useful for noisy chaotic systems.

Outline of the Technique

We start by assuming that there is a known set of data points $\mathbf{x}^{(n)}, n = 0, 1, 2, 3, \ldots, m$, a known model $\mathbf{x}^{(n)} = \mathbf{f}(\mathbf{x}^{(n-1)}; \mathbf{p})$ with unknown values of the parameters $\mathbf{p}$. Our goal is to figure out what the value of these parameters should be in our model to best match the data.

For example, consider the Henon map, which has two parameters $\mathbf{p} = \{a, b\}$:

$$\begin{aligned} x_1^{(n)} &= f_1 = 1 + x_2^{(n-1)} - a(x_1^{(n-1)})^2 \\ x_2^{(n)} &= f_2 = bx_1^{(n-1)} \end{aligned} \tag{1}$$

We can create a data set from the Henon map for specific initial conditions and specific values of parameters a, b using *Mathematica* (in applications, you would of course have the data but not the model parameters to begin with–we do this to create a dummy data set for experimentation). Once we have the data points, we could recover the values of the parameters by picking two sets of points and simply inverting Eqs. (1). *Mathematica* can do this for m different pairs of points:

```
lista = listb = {};
Do[
 eq1 = x1[n] == 1 + x2[n - 1] - a x1[n - 1]^2;
 eq2 = x2[n] == b x1[n - 1];
 sol = Solve[{eq1, eq2}, {a, b}];
 lista = Append[lista, sol[[1, 1, 2]]];
 listb = Append[listb, sol[[1, 2, 2]]];
, {n, 1, m}]
```

For all pairs, we will (not surprisingly) get the same values for a, b up to round off error (see left hand graph in Figure 1). If we try this same procedure with a noisy series, it does not work as well. Adding 1% random noise between $[0, 1]$ to the Henon map and proceeding as above, we get the result in the center of Figure 1. This shows that a simple inversion of Eqs. (1) will not work for noisy chaotic data sets.

To work with noisy chaotic maps the process outlined in Reference [1] can be used. It involves creating a new map by introducing new parameters k_1 and k_2. For the Henon map this new series would look like

$$\begin{aligned} y_1^{(n)} &= f_1 = 1 + y_2^{(n-1)} - a(y_1^{(n-1)})^2 + k_1 \\ y_2^{(n)} &= f_2 = by_1^{(n-1)} + k_2 \end{aligned}$$

The new modified map has a stable fixed point solution (the original map had an unstable fixed point), and since we have only added a constant the Jacobian for the modified map and the Henon map are identical. We now need to determine four unknown parameters a, b, k_1 and k_2.

This is done by solving the equations (pick any set of four $z_i^{(0)}, z_i^{(1)}, z_i^{(2)}, z_i^{(3)}$ from an arbitrary interval of time to get four equations):

$$c_i^{(n+1)} - c_i^{(n)} = z_i^{(n)} - z_i^{(n+1)} \tag{2}$$

where

$$z_i^{(n)} = x_i^{(n)}$$
$$c_i^{(n)} = k_i + \sum_{j=1}^{2} c_j^{(n-1)} \left.\frac{\partial f_i}{\partial x_j}\right|_{\mathbf{z}^{(n-1)}} + \frac{1}{2!}\sum_{j=1}^{2}\sum_{k=1}^{2} c_j^{(n-1)} c_k^{(n-1)} \left.\frac{\partial^2 f_i}{\partial x_j \partial x_k}\right|_{\mathbf{z}^{(n-1)}}$$

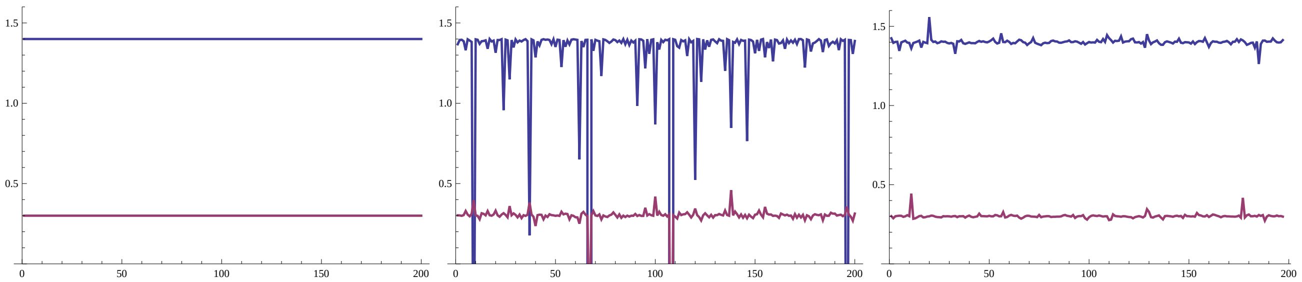

Figure 1: Determining values of a (blue) and b (red) given the data set $(x_1^{(n)}, x_2^{(n)})$. In this case, $a = 1.4$ and $b = 0.3$. Left: no noise, trivial inversion of Eq. (1). Center: noise, trivial inversion of Eq. (1). Right: noise, using Eq. (2).

Further Questions

- Are there some real world data that this could be applied to?
- Is there any benefit to keeping more terms in the Taylor series for $c_i^{(n)}$?
- How useful is this for determining the model $\mathbf{f}$?
- Is there anything interesting to say about relationship between the data sets $\mathbf{x}^{(n)}$ (chaotic) and $\mathbf{y}^{(n)}$ (stable)?
- What other techniques are there to estimate parameters in models? How does this fit in with those?

Prerequisites

- For this project, you should have completed at least Calculus II, since the process relies on a multivariable Taylor series.

What You Would Do

- You would start by reading Reference [1] and filling in all the details of the computation above.
- You will learn some of the theory behind discrete dynamical systems, and possibly some numerical methods.
- You would answer at least one of the *Further Questions*, or other questions that arose while you are filling in the details.
- You would finally create a poster of your results or give a talk at the URS.

References

[1] P. Palaniyandi and M. Lakshmanan *Estimation of system parameters in discrete dynamical systems from time series*, Physics Letters A **342** (2005), 134–139.